Foundations of Quantum Computing: Assignment 1

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Question 1

Consider the single-qubit unitary transformations:

$$Z|0\rangle = |0\rangle, \qquad Z|1\rangle = -|1\rangle,$$
$$H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \qquad H|1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle, \tag{1}$$
$$\mathrm{CX}|0,t\rangle = |0,t\rangle, \qquad \mathrm{CX}|1,t\rangle = |1,1-t\rangle.$$

Compute the state-vector:

$$Z_1 Z_2\, \mathrm{CX}_{2,1}\mathrm{CX}_{2,3} H_2 |0,0,0\rangle$$

Solution:

We start with the initial three-qubit state. Let's define ψ_0 and any following operation as ψ_i:

$$|\psi_0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 = |0,0,0\rangle.$$

Now we start right to left doing the maths.We apply the Hadamard gate on qubit 2:

$$|\psi_1\rangle = (I_1 \otimes H_2 \otimes I_3)|\psi_0\rangle = |0\rangle_1 \otimes \frac{1}{\sqrt{2}}(|0\rangle_2 + |1\rangle_2) \otimes |0\rangle_3 = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |0,1,0\rangle).$$

LEt's now make the $CX_{2,3}$ with qubit 2 as control and qubit 3 as target. We can remember that rule is: if the control qubit is $|1\rangle$, flip the target; if it is $|0\rangle$, do nothing. So then we have for the term $|0,0,0\rangle$, qubit 2 is $|0\rangle$, so qubit 3 remains $|0\rangle$. And for the term $|0,1,0\rangle$, qubit 2 is $|1\rangle$, so qubit 3 is flipped from $|0\rangle$ to $|1\rangle$. So then:

$$|\psi_2\rangle = \mathrm{CX}_{2,3}|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |0,1,1\rangle).$$

For the next $\mathrm{CX}_{2,1}$ with qubit 2 as control and qubit 1 as target. Again, if the control is $|1\rangle$ then flip the target. For $|0,0,0\rangle$: qubit 2 is $|0\rangle$, so qubit 1 remains $|0\rangle$. And for $|0,1,1\rangle$, qubit 2 is $|1\rangle$, so qubit 1 is flipped from $|0\rangle$ to $|1\rangle$.

$$|\psi_3\rangle = \mathrm{CX}_{2,1}\,|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle).$$

At this moment we can apply the Z gate on qubit 2. Recall: $Z\,|0\rangle = |0\rangle$ and $Z\,|1\rangle = -\,|1\rangle$. So then for $|0,0,0\rangle$ qubit 2 is $|0\rangle$, so the state remains unchanged. And for $|1,1,1\rangle$ qubit 2 is $|1\rangle$, so:

$$|\psi_4\rangle = (I_1 \otimes Z_2 \otimes I_3)\,|\psi_3\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle - |1,1,1\rangle).$$

Finally we apply the final Z gate on qubit 1. We know that, $Z\,|0\rangle = |0\rangle$ and $Z\,|1\rangle = -\,|1\rangle$. For $|0,0,0\rangle$, qubit 1 is $|0\rangle$, it remains unchanged. And for $-\,|1,1,1\rangle$, qubit 1 is $|1\rangle$, so applying all:

$$|\psi_5\rangle = (Z_1 \otimes I_2 \otimes I_3)\,|\psi_4\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle).$$

$$\boxed{|\psi_{\text{final}}\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)}.$$

Question 2

Consider the following four orthogonal two-qubit states:

$$\begin{aligned} |\Phi^+\rangle &= \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\,, & |\,\Psi^+\rangle &= \tfrac{1}{\sqrt{2}}|01\rangle + \tfrac{1}{\sqrt{2}}|10\rangle\,, \\ |\Phi^-\rangle &= \tfrac{1}{\sqrt{2}}|00\rangle - \tfrac{1}{\sqrt{2}}|11\rangle\,, & |\,\Psi^-\rangle &= \tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|10\rangle\,. \end{aligned} \tag{2}$$

Compute the state-vector

$$H_2\Big(\tfrac{1}{2}|\Phi^+\rangle + \tfrac{1}{2}|\Phi^-\rangle + \tfrac{1}{2}|\Psi^+\rangle + \tfrac{1}{2}|\Psi^-\Big\rangle)$$

Solution: We can write the state explicitly as:

$$|\psi\rangle = \frac{1}{2}\Big(|\Phi^+\rangle + |\Phi^-\rangle + |\Psi^+\rangle + |\Psi^-\rangle\Big).$$

And substituting the definitions of the states indicated in the exersice:

$$|\psi\rangle = \frac{1}{2}\left[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) + \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) + \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) + \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\right].$$

We can simplify the calculation as:

$$|\psi\rangle = \frac{1}{2\sqrt{2}}\Big[2|00\rangle + 2|01\rangle\Big] = \frac{1}{\sqrt{2}}\Big(|00\rangle + |01\rangle\Big).$$

Now we can apply the Hadamard transform on the second qubit. The Hadamard transformation acts on a single qubit as follows: $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. We write the state $|\psi\rangle$ in tensor product form:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle\Big) = |0\rangle \otimes \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big).$$

Apply the Hadamard gate H_2 on the second qubit:

$$H_2|\psi\rangle = |0\rangle \otimes H\Big(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\Big).$$

Recognizing that $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = H|0\rangle$. and the fact that H is self-inverse (i.e., $H^2 = I$), we have: $H\Big(H|0\rangle\Big) = |0\rangle$.

$$H_2|\psi\rangle = |0\rangle \otimes |0\rangle = |00\rangle.$$

The we can conhclude that:

$$\boxed{H_2\Big(\tfrac{1}{2}|\Phi^+\rangle + \tfrac{1}{2}|\Phi^-\rangle + \tfrac{1}{2}|\Psi^+\rangle + \tfrac{1}{2}|\Psi^-\rangle\Big) = |00\rangle.}$$

Question 3

Consider the state

$$|\Psi^-\rangle = \tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|10\rangle$$

Suppose that we perform a single-qubit measurement on the first qubit in one of the following bases. What is the state of the second qubit after measurement, if we measure the first qubit:

in the $|0\rangle, |1\rangle$ basis, and obtain the outcome $|0\rangle$?

Solution:

We can start writing the state in tensor product form:

$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$.

As indicated in the exercise if we get a measurement on the first qubit in the $\{|0\rangle, |1\rangle\}$ basis to be $|0\rangle$. To obtain the state after measurement, we apply the projection operator: $P_0 = |0\rangle\langle 0| \otimes I$ to $|\Psi^-\rangle$ and then we must renormalize:

$$\begin{aligned} P_0|\Psi^-\rangle &= (|0\rangle\langle 0| \otimes I)\left[\frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\Big)\right] \\ &= \frac{1}{\sqrt{2}}\Big(|0\rangle\langle 0|0\rangle \otimes |1\rangle - |0\rangle\langle 0|1\rangle \otimes |0\rangle\Big). \end{aligned}$$

Noticing that: $\langle 0|0\rangle = 1$ and $\langle 0|1\rangle = 0$, the projected state becomes:

$$P_0|\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big(1 \cdot |0\rangle \otimes |1\rangle - 0 \cdot |0\rangle \otimes |0\rangle\Big) = \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |1\rangle\Big).$$

We must renormalize the projected state. The norm of the state $\frac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle)$ is:

$$\left\| \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle) \right\| = \frac{1}{\sqrt{2}}.$$

After normalization, the state becomes:

$$|\psi\rangle_{\text{post}} = |0\rangle \otimes |1\rangle.$$

Since the measurement outcome on the first qubit is $|0\rangle$, the state of the second qubit after measurement is $|1\rangle$:

$$\boxed{|\psi_2\rangle = |1\rangle.}$$

Question 4

Consider the state

$$|\Psi^-\rangle = \tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|10\rangle.$$

Suppose that we perform a single-qubit measurement on the first qubit in one of the following bases. What is the state of the second qubit after measurement, if we measure the first qubit:

in the basis of the states $|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$ and $|-\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big)$, and obtain the outcome $|-\rangle$?

Solution: We can start expressing the computational basis states $|0\rangle$ and $|1\rangle$ in terms of the measurement basis $|+\rangle$ and $|-\rangle$:

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

Now rewriting $|\Psi^-\rangle$ using these expressions to have it in the desired basis:

$$\begin{aligned}
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}|0\rangle_1|1\rangle_2 - \frac{1}{\sqrt{2}}|1\rangle_1|0\rangle_2 \\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|+\rangle_1 + |-\rangle_1)|1\rangle_2\right] - \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|+\rangle_1 - |-\rangle_1)|0\rangle_2\right] \\
&= \frac{1}{2}\Big[(|+\rangle_1 + |-\rangle_1)|1\rangle_2 - (|+\rangle_1 - |-\rangle_1)|0\rangle_2\Big].
\end{aligned}$$

Expanding and grouping terms by the state of the first qubit:

$$\begin{aligned}
|\Psi^-\rangle &= \frac{1}{2}\Big[|+\rangle_1|1\rangle_2 + |-\rangle_1|1\rangle_2 - |+\rangle_1|0\rangle_2 + |-\rangle_1|0\rangle_2\Big] \\
&= \frac{1}{2}\Big[|+\rangle_1(|1\rangle_2 - |0\rangle_2) + |-\rangle_1(|1\rangle_2 + |0\rangle_2)\Big].
\end{aligned}$$

When the first qubit is measured to be $|-\rangle_1$, the (unnormalized) post-measurement state of the system is given by the term with $|-\rangle_1$:

$$|\psi\rangle_{\text{post}} = \frac{1}{2}|-\rangle_1(|1\rangle_2 + |0\rangle_2).$$

Since the measurement projects the first qubit onto $|-\rangle_1$, the state of the second qubit (unnormalized) is:

$$|\psi_2\rangle_{\text{un}} = \frac{1}{2}(|0\rangle_2 + |1\rangle_2).$$

Now normalizing the state of the second qubit. First we compute the norm of $|\psi_2\rangle_{\text{un}}$:

$$\begin{aligned}\left\|\frac{1}{2}(|0\rangle + |1\rangle)\right\| &= \frac{1}{2}\sqrt{\langle 0+1|0+1\rangle} \\ &= \frac{1}{2}\sqrt{\langle 0|0\rangle + \langle 0|1\rangle + \langle 1|0\rangle + \langle 1|1\rangle} \\ &= \frac{1}{2}\sqrt{1+0+0+1} \\ &= \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}.\end{aligned}$$

So it is enough to multiply the unnormalized state by the reciprocal of the norm (i.e., $\sqrt{2}$) to obtain the normalized state:

$$|\psi_2\rangle = \sqrt{2}\cdot\frac{1}{2}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Now if we remeber that:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle.$$

After measuring the first qubit in the $|+\rangle$, $|-\rangle$ basis and obtaining the outcome $|-\rangle$, the state of the second qubit is:

$$\boxed{|\psi_2\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).}$$

Question 5

Consider the state

$$|\Psi^-\rangle = \tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|10\rangle.$$

Suppose that we perform a single-qubit measurement on the first qubit in one of the following bases. What is the state of the second qubit after measurement, if we measure the first qubit:

in the basis of the states $|+i\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + i|1\rangle\big)$ and $|-i\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - i|1\rangle\big)$, and obtain the outcome $|+i\rangle$?

Solution:

As in the previous exersice we start by expressing the computational basis states $|0\rangle$ and $|1\rangle$ in terms of the measurement basis $|+i\rangle$ and $|-i\rangle$. We have: $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. Adding these equations gives: $|+i\rangle + |-i\rangle = \frac{1}{\sqrt{2}}(2|0\rangle)$, so $|0\rangle = \frac{1}{\sqrt{2}}(|+i\rangle + |-i\rangle)$.

Subtracting the second from the first: $|+i\rangle - |-i\rangle = \frac{1}{\sqrt{2}}(2i|1\rangle)$, so $|1\rangle = \frac{1}{\sqrt{2}\,(2i)} \cdot (2i|1\rangle) = \frac{1}{\sqrt{2}} \cdot \frac{1}{i}(|+i\rangle - |-i\rangle)$.

Since $\frac{1}{i} = -i$, we obtain: $|1\rangle = -\frac{i}{\sqrt{2}}(|+i\rangle - |-i\rangle)$. Now we rewrite $|\Psi^-\rangle$ in terms of $|+i\rangle$ and $|-i\rangle$:

$$\begin{aligned}
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}\,|0\rangle_1|1\rangle_2 - \frac{1}{\sqrt{2}}\,|1\rangle_1|0\rangle_2 \\
&= \frac{1}{\sqrt{2}}\left[\left(\frac{1}{\sqrt{2}}(|+i\rangle_1 + |-i\rangle_1)\right)|1\rangle_2\right] - \frac{1}{\sqrt{2}}\left[\left(-\frac{i}{\sqrt{2}}(|+i\rangle_1 - |-i\rangle_1)\right)|0\rangle_2\right] \\
&= \frac{1}{2}\Big[(|+i\rangle_1 + |-i\rangle_1)|1\rangle_2 + i(|+i\rangle_1 - |-i\rangle_1)|0\rangle_2\Big].
\end{aligned}$$

Let's group the terms corresponding to the terms with $|+i\rangle_1$ and $|-i\rangle_1$:

$$|\Psi^-\rangle = \frac{1}{2}\Big[|+i\rangle_1\Big(|1\rangle_2 + i|0\rangle_2\Big) + |-i\rangle_1\Big(|1\rangle_2 - i|0\rangle_2\Big)\Big].$$

Since the measurement outcome on the first qubit is $|+i\rangle_1$, we project onto this state. The (unnormalized) post-measurement state is given by the corresponding term:

$$|\psi\rangle_{\text{post}} = \frac{1}{2}\,|+i\rangle_1\Big(|1\rangle_2 + i|0\rangle_2\Big).$$

Thus, the unnormalized state of the second qubit is:

$$|\psi_2\rangle_{\text{un}} = |1\rangle + i|0\rangle.$$

we are going to compute the norm so we can normalize:

$$\begin{aligned}
\||\psi_2\rangle_{\text{un}}\|^2 &= \langle\psi_2|\psi_2\rangle_{\text{un}} = \langle 1|1\rangle + |i|^2\langle 0|0\rangle \\
&= 1 + 1 = 2.
\end{aligned}$$

Therefore, the normalized state is:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big(|1\rangle + i|0\rangle\Big).$$

After measuring the first qubit in the $|+i\rangle$, $|-i\rangle$ basis and obtaining the outcome $|+i\rangle$, the state of the second qubit is:

$$\boxed{|\psi_2\rangle = \frac{1}{\sqrt{2}}\Big(|1\rangle + i|0\rangle\Big).}$$

Question 6

Consider the single-qubit unitary transformations:

$$R_y(\theta) \;=\; \mathrm{e}^{i\theta/2}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{bmatrix}\;, \qquad R_z(\phi) \;=\; \begin{bmatrix}1 & 0\\ 0 & \mathrm{e}^{i\phi}\end{bmatrix}\;. \tag{3}$$

Consider the single-qubit unitary transformation $U(\theta,\phi)$, defined as a function of two angular parameters θ and ϕ, which would be the result of an $R_z(\phi)$ operation, followed by an $R_y(\theta)$ operation, followed by an $R_z(-\phi)$ operation.

Write the matrix for $U(\theta,\phi)$, as a function of the angular parameters θ and ϕ.

Solution:

As defined $U(\theta,\phi)$ is the product of the three matrices:

$$U(\theta,\phi) = R_z(-\phi)\,R_y(\theta)\,R_z(\phi).$$

Substituting the definitions and doing first the the product $R_y(\theta)\;R_z(\phi)$:

$$R_y(\theta)R_z(\phi) = e^{i\theta/2}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\phi}\end{pmatrix} = e^{i\theta/2}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\,e^{i\phi}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\,e^{i\phi}\end{pmatrix}.$$

Now multiplying on the left by $R_z(-\phi)$:

$$\begin{aligned}U(\theta,\phi) &= R_z(-\phi)\left(e^{i\theta/2}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\,e^{i\phi}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\,e^{i\phi}\end{pmatrix}\right)\\ &= e^{i\theta/2}\begin{pmatrix}1 & 0\\ 0 & e^{-i\phi}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\,e^{i\phi}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\,e^{i\phi}\end{pmatrix}.\end{aligned}$$

Performing the matrix multiplication:

$$\begin{aligned}U(\theta,\phi) &= e^{i\theta/2}\begin{pmatrix}1\cdot\cos\frac{\theta}{2} + 0\cdot\sin\frac{\theta}{2} & 1\cdot\left(-\sin\frac{\theta}{2}\,e^{i\phi}\right) + 0\cdot\cos\frac{\theta}{2}\,e^{i\phi}\\ 0\cdot\cos\frac{\theta}{2} + e^{-i\phi}\cdot\sin\frac{\theta}{2} & 0\cdot\left(-\sin\frac{\theta}{2}\,e^{i\phi}\right) + e^{-i\phi}\cdot\cos\frac{\theta}{2}\,e^{i\phi}\end{pmatrix}\\ &= e^{i\theta/2}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\,e^{i\phi}\\ \sin\frac{\theta}{2}\,e^{-i\phi} & \cos\frac{\theta}{2}\end{pmatrix}.\end{aligned}$$

So to conclude:

$$\boxed{U(\theta,\phi) = e^{i\theta/2}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\,e^{i\phi}\\ \sin\frac{\theta}{2}\,e^{-i\phi} & \cos\frac{\theta}{2}\end{pmatrix}.}$$

Question 7

Consider the single-qubit unitary transformations:

$$R_y(\theta) \;=\; \mathrm{e}^{i\theta/2}\begin{bmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{bmatrix}, \qquad R_z(\phi) \;=\; \begin{bmatrix}1 & 0\\ 0 & \mathrm{e}^{i\phi}\end{bmatrix}. \tag{4}$$

Consider the single-qubit unitary transformation $U(\theta, \phi)$, defined as a function of two angular parameters θ and ϕ, which would be the result of an $R_z(\phi)$ operation, followed by an $R_y(\theta)$ operation, followed by an $R_z(-\phi)$ operation.

Let $V = U(1, 1)$. Write the matrix for the operation CV. (Do not evaluate the numerical values of any trigonometric expressions).

Solution: We begin by stating from the last exercise that U is:

$$U(\theta, \phi) = e^{i\theta/2}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\,e^{i\phi}\\ \sin\frac{\theta}{2}\,e^{-i\phi} & \cos\frac{\theta}{2}\end{pmatrix}.$$

Now we set $(\theta = 1)$ and $\phi = 1$ to define:

$$V = U(1, 1) = e^{i/2}\begin{pmatrix}\cos\frac{1}{2} & -\sin\frac{1}{2}\,e^{i}\\ \sin\frac{1}{2}\,e^{-i} & \cos\frac{1}{2}\end{pmatrix}.$$

The controlled-V gate, denoted as CV, is a two-qubit operation that applies the identity operator to the target qubit when the control qubit is in the state $|0\rangle$ and applies V when the control qubit is in the state $|1\rangle$.

In the standard computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the matrix representation is block-diagonal:

$$\mathrm{C}V = \begin{pmatrix}I_2 & 0\\ 0 & V\end{pmatrix},$$

where I_2 is the 2×2 identity matrix. Substituting the expression for V, we obtain:

$$\mathrm{C}V = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & e^{i/2}\cos\frac{1}{2} & -e^{i/2}\sin\frac{1}{2}\,e^{i}\\ 0 & 0 & e^{i/2}\sin\frac{1}{2}\,e^{-i} & e^{i/2}\cos\frac{1}{2}\end{pmatrix}.$$

This is the full 4×4 matrix representation of the controlled-V gate. So the answer is:

$$\boxed{\mathrm{C}V = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & e^{i/2}\cos\frac{1}{2} & -e^{i/2}\sin\frac{1}{2}\,e^{i}\\ 0 & 0 & e^{i/2}\sin\frac{1}{2}\,e^{-i} & e^{i/2}\cos\frac{1}{2}\end{pmatrix}.}$$

Question 8

Consider the four following orthogonal two-qubit states:

$$\begin{aligned} |\Phi^+\rangle &= \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\,, & |\,\Psi^+\rangle &= \tfrac{1}{\sqrt{2}}|01\rangle + \tfrac{1}{\sqrt{2}}|10\rangle\,, \\ |\Phi^-\rangle &= \tfrac{1}{\sqrt{2}}|00\rangle - \tfrac{1}{\sqrt{2}}|11\rangle\,, & |\,\Psi^-\rangle &= \tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|10\rangle\,. \end{aligned} \tag{5}$$

Compute the probability of obtaining each of these four states, if we measured two qubits in the state $|{+i},{+i}\rangle$ with respect to this basis, where $|{+i}\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + i|1\rangle\big)$.

Solution:

We start by writing the state $|{+i},{+i}\rangle$ in the tensor format:

$$|{+i},{+i}\rangle = |{+i}\rangle \otimes |{+i}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) = \frac{1}{2}(|0\rangle + i\,|1\rangle) \otimes (|0\rangle + i\,|1\rangle).$$

Expanding the tensor product, we have:

$$|{+i},{+i}\rangle = \frac{1}{2}\big(\,|00\rangle + i\,|01\rangle + i\,|10\rangle + i^2\,|11\rangle\,\big).$$

Since $i^2 = -1$, this becomes:

$$|{+i},{+i}\rangle = \frac{1}{2}(|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle).$$

We recall the four two bit states:

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\ |\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ |\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ |\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned}$$

We need to compute the amplitude state, and then square the modulus to get the probabilities. So for each of the indicated states:

(a) For $|\Phi^+\rangle$

$$\langle \Phi^+ | + i, {+i}\rangle = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\frac{1}{2}(|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle).$$

Only the terms with $|00\rangle$ and $|11\rangle$ contribute since any other combination is zero as they are an orthogonal basis:

$$\langle \Phi^+ | +i, +i \rangle = \frac{1}{2\sqrt{2}}[\langle 00|\,00\rangle - \langle 11|\,11\rangle] = \frac{1}{2\sqrt{2}}[1-1] = 0.$$

Thus,

$$P(\Phi^+) = \left|\langle \Phi^+ | +i, +i \rangle\right|^2 = 0.$$

(b) For $|\Phi^-\rangle$

$$\langle \Phi^- | +i, +i \rangle = \frac{1}{\sqrt{2}}(\langle 00| - \langle 11|)\frac{1}{2}(|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle).$$

Again, only the $|00\rangle$ and $|11\rangle$ terms contribute:

$$\langle \Phi^- | +i, +i \rangle = \frac{1}{2\sqrt{2}}[\langle 00|\,00\rangle + \langle 11|\,11\rangle] = \frac{1}{2\sqrt{2}}[1+1] = \frac{1}{\sqrt{2}}.$$

Therefore,

$$P(\Phi^-) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

(c) For $|\Psi^+\rangle$

$$\langle \Psi^+ | +i, +i \rangle = \frac{1}{\sqrt{2}}(\langle 01| + \langle 10|)\frac{1}{2}(|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle).$$

Only the $|01\rangle$ and $|10\rangle$ terms contribute:

$$\langle \Psi^+ | +i, +i \rangle = \frac{1}{2\sqrt{2}}[i\,\langle 01|\,01\rangle + i\,\langle 10|\,10\rangle] = \frac{1}{2\sqrt{2}}[i+i] = \frac{i}{\sqrt{2}}.$$

Hence,

$$P(\Psi^+) = \left|\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

(d) For $|\Psi^-\rangle$

$$\langle \Psi^- | +i, +i \rangle = \frac{1}{\sqrt{2}}(\langle 01| - \langle 10|)\frac{1}{2}(|00\rangle + i\,|01\rangle + i\,|10\rangle - |11\rangle).$$

The contributions from $|01\rangle$ and $|10\rangle$ cancel:

$$\langle \Psi^- | +i, +i \rangle = \frac{1}{2\sqrt{2}}[i-i] = 0.$$

Therefore,

$$P(\Psi^-) = 0.$$

Summarizing the results are:

$$\boxed{\begin{aligned} P(\Phi^+) &= 0, \\ P(\Phi^-) &= \frac{1}{2}, \\ P(\Psi^+) &= \frac{1}{2}, \\ P(\Psi^-) &= 0. \end{aligned}}$$

Question 9

Consider the following operation:

$$\begin{aligned} \mathrm{CCX}\,|0,0,t\rangle &= |0,0,t\rangle, & \mathrm{CCX}\,|1,0,t\rangle &= |1,0,t\rangle, \\ \mathrm{CCX}\,|0,1,t\rangle &= |0,1,t\rangle, & \mathrm{CCX}\,|1,1,t\rangle &= |1,1,1-t\rangle. \end{aligned}$$

Consider the state $|\psi_0\rangle = \mathrm{CCX}\,|+,+,0\rangle$ where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Suppose that we measure the third qubit of $|\psi\rangle$ in the computational basis. What is the probability of obtaining the outcome $|0\rangle$?

Solution:

Lets rewrite the initial three-qubit state:

$$|+,+,0\rangle = |+\rangle \otimes |+\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big) \otimes \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big) \otimes |0\rangle.$$

Expanding the tensor product, we have:

$$|+,+,0\rangle = \frac{1}{2}\Big(|0,0,0\rangle + |0,1,0\rangle + |1,0,0\rangle + |1,1,0\rangle\Big).$$

Apply the CCX gate:

$$\begin{aligned} \mathrm{CCX}\,|0,0,0\rangle &= |0,0,0\rangle, \\ \mathrm{CCX}\,|0,1,0\rangle &= |0,1,0\rangle, \\ \mathrm{CCX}\,|1,0,0\rangle &= |1,0,0\rangle, \\ \mathrm{CCX}\,|1,1,0\rangle &= |1,1,1\rangle. \end{aligned}$$

Thus, after applying the CCX gate:

$$|\psi_0\rangle = \mathrm{CCX}\,|+,+,0\rangle = \frac{1}{2}\Big(|0,0,0\rangle + |0,1,0\rangle + |1,0,0\rangle + |1,1,1\rangle\Big).$$

The terms in $|\psi_0\rangle$ with the third qubit in the state $|0\rangle$ are:

$$|0,0,0\rangle, \quad |0,1,0\rangle, \quad |1,0,0\rangle.$$

Each of these terms has an amplitude of $\frac{1}{2}$. We know that the probability $P(|0\rangle)$ of measuring the third qubit in the state $|0\rangle$ is the sum of the squares of the moduli of these amplitudes:

$$P(|0\rangle) = \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

The answer is:

$$\boxed{P(|0\rangle) = \frac{3}{4}}$$

Question 10

Consider the following operation:

$$\begin{aligned} \text{CCX}\,|0,0,t\rangle &= |0,0,t\rangle, & \text{CCX}\,|1,0,t\rangle &= |1,0,t\rangle, \\ \text{CCX}\,|0,1,t\rangle &= |0,1,t\rangle, & \text{CCX}\,|1,1,t\rangle &= |1,1,1-t\rangle. \end{aligned}$$

Consider the state $|\psi_0\rangle = \text{CCX}\,|+,+,0\rangle$ where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Suppose that we measure the third qubit of $|\ \psi\rangle$ in the computational basis and obtain the outcome $|\ 0\rangle$. Conditioned on this outcome, what is the probability of obtaining the outcome $|\ 10\rangle$ if we measure the first two qubits?

Solution:

Starting with the initial qubit state:

$$|+,+,0\rangle = |+\rangle \otimes |+\rangle \otimes |0\rangle = \frac{1}{2}\Big(|0,0,0\rangle + |0,1,0\rangle + |1,0,0\rangle + |1,1,0\rangle\Big).$$

Remember how the CCX operates:

$$\begin{aligned} \text{CCX}\,|0,0,0\rangle &= |0,0,0\rangle, \\ \text{CCX}\,|0,1,0\rangle &= |0,1,0\rangle, \\ \text{CCX}\,|1,0,0\rangle &= |1,0,0\rangle, \\ \text{CCX}\,|1,1,0\rangle &= |1,1,1\rangle. \end{aligned}$$

Thus, after applying it, the state becomes:

$$|\psi_0\rangle = \text{CCX}\,|+,+,0\rangle = \frac{1}{2}\Big(|0,0,0\rangle + |0,1,0\rangle + |1,0,0\rangle + |1,1,1\rangle\Big).$$

The terms with the third qubit in state $|0\rangle$ are:

$$|0,0,0\rangle, \quad |0,1,0\rangle, \quad |1,0,0\rangle.$$

The unnormalized post-measurement state (conditioned on obtaining $|0\rangle$) is:

$$|\tilde{\psi}\rangle = \frac{1}{2}\Big(|0,0,0\rangle + |0,1,0\rangle + |1,0,0\rangle\Big).$$

The probability of measuring $|0\rangle$ in the third qubit is given by the sum of the squared magnitudes of the amplitudes:

$$P(|0\rangle_3) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

The normalized state after obtaining $|0\rangle$ for the third qubit is:

$$|\psi_{\text{post}}\rangle = \frac{|\tilde{\psi}\rangle}{\sqrt{P(|0\rangle_3)}} = \frac{1}{\sqrt{3}}\Big(|0,0,0\rangle + |0,1,0\rangle + |1,0,0\rangle\Big).$$

We wish to find the probability of obtaining the outcome $|10\rangle$ for the first two qubits. In the state $|\psi_{\text{post}}\rangle$, the term corresponding to the first two qubits being $|10\rangle$ is $|1,0,0\rangle$ which has an amplitude of $\frac{1}{\sqrt{3}}$.

Thus, the probability (conditioned on having obtained $|0\rangle$ for the third qubit) is:

$$\boxed{P(|10\rangle \mid |0\rangle_3) = \left|\frac{1}{\sqrt{3}}\right|^2 = \frac{1}{3}}$$